On time evolution of quantum black holes

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Abstract

The time evolution of black holes involves both the canonical equations of quantum gravity and the statistical mechanics of Hawking radiation, neither of which contains a time variable. In order to introduce the time, we apply the semiclassical approximation to the Hamiltonian constraint on the apparent horizon and show that, when the backreaction is included, it suggests the existence of a long-living remnant, similarly to what is obtained in the microcanonical picture for the Hawking radiation.

Key words: black holes, Hawking effect, microcanonical ensemble, semiclassical approximation, quantum gravity
PACS: 04.70.-s, 04.70.Dy, 04.60.-m

A longstanding issue in theoretical physics is how to quantize Einstein gravity. Since in the theory there are constraints whose algebra contains structure functions (which depend on the space-time coordinates), one can conceive (inequivalent [1]) manners of lifting the Hamiltonian and momentum constraints to quantum equations, thus hindering the formal solution to the problem. On a more physical ground, one might notice that it is relevant to have a quantum theory of gravity at our disposal only for systems with very strong (Planck size) gravitational fields, such as those one expects in the early stages of the Universe. Because of Hawking's discovery of black hole evaporation [2], one also expect that quantum (or semiclassical) gravity plays a role in determining the dynamics of the late stages in the life of collapsed objects.

In a general situation, one has to deal with the infinite number of degrees of freedom of the gravitational field configurations which are physically distinguished only modulo coordinate transformations. This gives the constraints the form of functional differential equations, for which there is little hope of finding general solutions, and one then tries to reduce the number of degrees

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of freedom from the onset. One way of implementing this program is the *back-ground field method* [3,4], in which one assumes a certain background metric manifold and does perturbation theory on it (although it turns out to be non-renormalizable). Another method is to impose a large (space-time) symmetry *prior* to quantization, resulting in a so called *mini-* (or *midi-*) *superspace* for a finite (or discrete) number of canonical variables [5,6]. In the latter approach to quantum gravity there is no time and one expects that a time variable can be identified by resorting to the semiclassical approximation [7].

Unfortunately, there are systems which do not allow for such a reduction, and (spherical) black holes fall among these cases, because one cannot find a finite set of variables in which to express both the Hamiltonian and the momentum constraints consistently over the whole space-time manifold [8]. Years ago, Tomimatsu [9] proposed an alternative scheme of reduction by considering the support of the constraints just near the apparent horizon and quantizing the residual degrees of freedom.

Hawking radiation can also be studied from the point of view of the statistical mechanics by assuming the horizon area $\mathcal{A}$ is a measure of the internal (microscopical) degeneracy σ of black holes according to [10] [2]

$$\sigma \sim \exp\left(\frac{\mathcal{A}}{4}\right) \ , \tag{1}$$

and thus determines the probability

$$P \simeq \frac{c}{\sigma} \ , \tag{2}$$

for Hawking particles to tunnel out through the horizon. When the space-time is asymptotically flat, it is known that the canonical ensemble is formally inconsistent and one should instead use the microcanonical ensemble in order to ensure global energy conservation in the system [11]. For large black holes the two descriptions are practically equivalent, but they then disagree when the mass becomes small (of order the Planck mass). Within this approach the detailed form of the metric (including the backreaction) affects the grey-body factor [13] and the constant c in Eq. (2) cannot be determined without a complete knowledge of the (non-local) physics near the horizon. In this respect, the microcanonical picture is complementary to the dynamical approach of Ref. [9].

The aim of this letter is precisely to study the consequences which can be derived from confronting the constraints of canonical (quantum) gravity lo-

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[2] This relation suggests that black holes are extended *p*-branes [11] and was also obtained in string theory [12].

calized on the apparent horizon as in Ref. [9] with the global microcanonical picture of the Hawking effect, with a particular emphasis on the role played by the backreaction of the emitted radiation.

Let us start from the dynamical approach. Tomimatsu [9] considered a conformally coupled scalar field ϕ in a four-dimensional spherically symmetric space-time with action [14]

$$S = \frac{1}{2} \int d^2x \sqrt{-g} \left[1 + g^{ab} \partial_a \varphi \partial_b \varphi + \frac{1}{2} R \varphi^2 - \varphi^2 g^{ab} \partial_a \phi \partial_b \phi \right] . \tag{3}$$

The four-dimensional metric has been written as

$$ds^2 = g_{ab} dx^a dx^b + \varphi^2 d\Omega^2 , \tag{4}$$

where $x^a = (t, r)$, $d\Omega^2$ is the line element of a unit two-sphere, $\varphi = \varphi(t, r)$ the radius of a two-sphere of area $4\pi\varphi^2$, $g_{ab} = g_{ab}(t, r)$ a two-dimensional metric and R its scalar curvature. Adopting the ADM decomposition of a generic two-dimensional metric [15]

$$g_{ab} = \begin{bmatrix} -N^2 + \dfrac{N_r^2}{\gamma} & N_r \\ N_r & \gamma \end{bmatrix} , \tag{5}$$

with N the lapse function and N_r the (radial) shift function, one obtains that the momentum H_r and the Hamiltonian H coincide [3] near the apparent horizon (defined by $g_{tt} = 0$), $H_r = H/\sqrt{2}$. This just leaves the Wheeler-DeWitt equation (Hamiltonian constraint) for the wavefunction $\Psi = \Psi(\phi, m)$,

$$\frac{1}{2} \left[\frac{\hat{\Pi}_\phi^2}{4\,m^2} - 2\,\hat{\Pi}_m + \frac{1}{2} \right] \Psi = 0 , \tag{6}$$

where m is the black hole mass (in units with $c = \hbar = G = 1$) as it appears in $g_{tt} = -1 + 2\,m/r$. The classical momenta are given by

$$\Pi_m = \frac{1}{2} \left(\dot{m} + \frac{1}{2} \right) , \quad \Pi_\phi = 4\,m^2\,\dot{\phi} , \tag{7}$$

and a dot denotes derivative with respect to the time t measured in the reference frame of a (distant) static observer. To summarize, the system near

[3] In a suitable gauge with $\gamma = 2$ and its conjugate momentum $\Pi_\gamma = \varphi/4$. Consequently γ is not dynamical [9].

the apparent horizon has two physical degrees of freedom: the Bondi mass m computed on the outer surface of the apparent horizon (of radius $\varphi = 2\,m$) and the scalar field ϕ which can be used to reproduce the outgoing Hawking radiation as we shall show below (see also Ref. [9]).

Eq. (6) can be solved exactly by separation of variables, and one finds out the set of solutions

$$\Psi(m,\phi) = \Psi_0 \, \exp\left\{ i\left(\frac{k^2}{8\,m} - \frac{m}{4} - k\,\phi\right)\right\} \ , \tag{8}$$

where Ψ_0 is a constant and k a continuous (complex) wave number. For k real one has "plane waves" corresponding to a static black hole, and for k imaginary Hawking's behaviour [2] is recovered [9]. Despite the simple form of the derivation, the above solution taken at face value is not very illuminating about the temporal evolution of the black hole, since there is no time dependence in the function in Eq. (8), and one needs, at least in principle, to build a suitable superposition of modes in such a way that a time variable can be defined (in terms of m and ϕ).

In practice, one resorts to the semiclassical approximation. This has been attempted already in Ref. [9], but here we want to employ the more systematic Born-Oppenheimer approach (see, for instance, Ref. [7,16]). First of all, we assume that the complete wavefunction factorizes into "gravitational" (ψ) and "matter" (χ) parts,

$$\Psi = \psi(m)\,\chi(\phi;m) \ , \tag{9}$$

from which we get, upon substituting into Eq. (6), the coupled equations

$$\begin{cases} \hat{\Pi}_m\,\tilde{\psi} = \langle\,\hat{H}_\phi\,\rangle\,\tilde{\psi} \\[2em] \hat{\Pi}_m\,\tilde{\chi} = \left[\hat{H}_\phi - \langle\,\hat{H}_\phi\,\rangle\right]\,\tilde{\chi} \ , \end{cases} \tag{10}$$

where we have introduced the rescaled functions

$$\tilde{\psi} = \psi\,e^{+i\int\langle\,\hat{\Pi}_m\,\rangle\,dm} \ , \qquad \tilde{\chi} = \chi\,e^{-i\int\langle\,\hat{\Pi}_m\,\rangle\,dm} \ , \tag{11}$$

and the scalar field Hamiltonian

$$H_\phi = \frac{1}{2}\left[\frac{\Pi_\phi^2}{4\,m^2} + \frac{1}{2}\right] \ . \tag{12}$$

In the above $\langle\,\hat{A}(m)\,\rangle \equiv \int d\phi\, \chi^*(\phi;m)\,\hat{A}\,\chi(\phi;m)$ for any operator $\hat{A}$. It is noticeable that Eqs. (10) do not contain terms corresponding to gravitational fluctuations of the form which is obtained when the momentum Π_m enters quadratically [16,17]. This signals the fact that, on restricting the support of the constraints just on the apparent horizon, one is actually freezing most of the quantum fluctuations for the system [18].

We now assume that the variable m behaves (almost) classically. Correspondingly, the gravitational wave function $\tilde{\psi}$ is a wave-packet peaked on a classical trajectory $m = m(t)$, thus it is well approximated by the WKB form

$$\tilde{\psi} \simeq \frac{e^{i\,S_{cl}[m]}}{\sqrt{\partial_m S_{cl}}} \ , \tag{13}$$

where the classical action S_{cl} in the exponent is evaluated along $m(t)$. The time variable t can then be naturally related to the mass m by

$$\frac{\partial}{\partial t} \equiv 2\left[i\,\frac{\partial}{\partial m}(\ln\tilde{\psi}) - \frac{1}{4}\right]\frac{\partial}{\partial m} = \dot{m}\,\frac{\partial}{\partial m} \ . \tag{14}$$

To leading order (in $\ell_p\,m_p = \hbar$) the first of Eqs. (10) then gives the Hamilton-Jacobi equation

$$\dot{m} = 2\left[\langle\,\hat{H}_\phi\,\rangle - \frac{1}{4}\right] = \frac{\langle\,\hat{\Pi}_\phi^2\,\rangle}{4\,m^2} \ , \tag{15}$$

where we used $\partial_m S_{cl} = \Pi_m$, and the second of Eqs. (10) is unaffected by the above approximation. If $\langle\,\hat{\Pi}_\phi^2\,\rangle = k^2$ and constant, we recover the result of Ref. [9]. In particular, for $k = i\,\kappa_2$ one has

$$\dot{m} = -\frac{\kappa_2^2}{4\,m^2} \ , \tag{16}$$

which is the well known Hawking's law of black hole decay [2] implying a finite time of evaporation (as measured by a static observer). The numerical coefficient κ_2 is arbitrary and its value could possibly be computed by considering the constraints over the complete space-time manifold.

Instead of assuming that the state of the scalar field remains a (momentum) eigenstate, we can use the second of Eqs. (10) to determine the evolution of the state of ϕ, thus including the backreaction of the change in mass onto the radiation. Upon introducing the time according to Eq. (14) and making use of Eq. (15), we get the Schrödinger-like equation

$$i \, \frac{\partial \chi_s}{\partial t} = -\frac{\dot{m}}{8 \, m^2} \, \hat{\Pi}_\phi^2 \, \chi_s \equiv \hat{\mathcal{H}}_\phi \, \chi_s \ , \tag{17}$$

where $\chi_s = \tilde{\chi} \, e^{-i \int \langle \hat{\mathcal{H}}_\phi \rangle \, dt}$.

Schrödinger equations with time-dependent parameters can be solved by introducing invariant operators [19], whose form is explicitly known for the generalized harmonic oscillator (see Ref. [20] for the details). In order to apply this method to Eq. (17) with an arbitrary $m = m(t)$, one must first "regularize" the Hamiltonian $\mathcal{H}_\phi$ by generating a non-vanishing "oscillator frequency". This can be done by introducing a (small) harmonic-oscillator-like potential [4],

$$\mathcal{H}_\phi \rightarrow \mathcal{H}_\phi + \frac{\epsilon}{2} \, \phi^2 \ , \tag{18}$$

where $\epsilon \rightarrow 0$ at the end of the computation. One then finds the spectrum $\{ | \, n, t \, \rangle \, | \, n \in \mathbb{N} \}$ of (exact) solutions to Eq. (17) and, on using Eq. (15),

$$\begin{aligned} \dot{m}^2 &= 2 \, \langle \, n, t \, | \, \hat{\mathcal{H}}_\phi \, | \, n, t \, \rangle \\ &= \left(n + \frac{1}{2} \right) \left[\epsilon \, x^2 - 4 \, m^2 \, \frac{\dot{x}^2}{\dot{m}} - \frac{\dot{m}}{4 \, m^2 \, x^2} \right] \ , \end{aligned} \tag{19}$$

where the auxiliary function $x = x(t)$ must satisfy

$$4 \, \frac{d}{dt} \left(m^2 \, \frac{\dot{x}}{\dot{m}} \right) - \epsilon \, x = \frac{\dot{m}}{4 \, m^2 \, x^3} \ , \tag{20}$$

with suitable initial conditions. The latter must be derived from some physical argument, as we shall attempt in the following. We further note that, for an evaporating black hole, $\dot{m} < 0$ and the right hand side of Eq. (19) is consistently positive.

We shall assume that, for large mass ($m \gg 1$), Hawking's behaviour (16) be recovered and determine the corresponding $x = x_H(t)$ by expanding in powers of $1/m$,

$$x_H = \sum_{i \geq 0} \frac{x_i}{m^i} \ , \tag{21}$$

where the coefficients $x_i = x_i(t)$. Inserting the sum (21) into Eqs. (19) and (20) yields, in the limit for $\epsilon \rightarrow 0$,

[4] The ϵ-term can be viewed as a small mass for the scalar particle (IR cut-off) or as a prescription to discretize the spectrum.
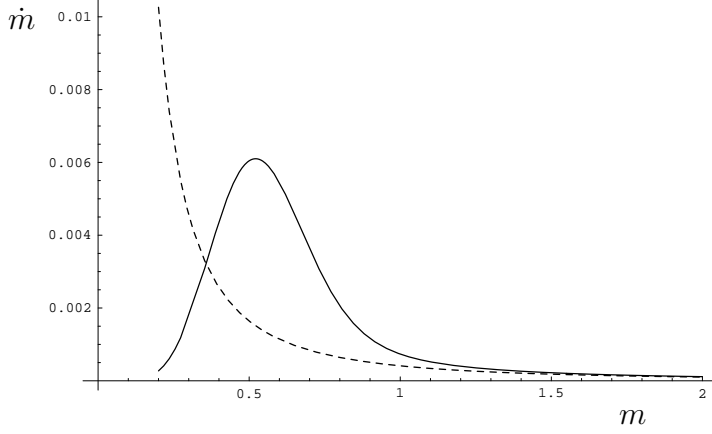
Fig. 1. Luminosity of a radiating black hole in the canonical (dashed line) and microcanonical (continuous line) pictures.

$$x_H = \sqrt{\frac{2\,n+1}{2\,\kappa_2}} + \mathcal{O}\left(\frac{1}{m^8}\right) \;, \tag{22}$$

and one can further check that Eq. (16) is a consistent solution to the system of Eqs. (19) and (20) to even higher orders (details will be given elsewhere [21]).

From the microcanonical description we know that the law (16) must be corrected for small masses. In fact the correct number density of Hawking quanta is better approximated by the microcanonical expression [11]

$$N(\omega) = \sum_{l=1}^{[[m/\omega]]} \frac{\exp\left[4\,\pi\,(m-l\,\omega)^2\right]}{\exp(4\,\pi\,m^2)} \;, \tag{23}$$

where $[[X]]$ denotes the integer part of X. In the limit $m \to \infty$, N equals the canonical ensemble number density (Planckian distribution). The decay rate (the opposite of the luminosity) for a black hole is then given by

$$\dot{m} = -\mathcal{A} \sum_{k} \int_0^\infty N_k\,(\omega)\,\Gamma_k(\omega)\,\omega^3\,d\omega \;, \tag{24}$$

where Γ is the grey-body factor and the sum is over particle species and angular momentum [22]. The comparisons between the luminosity and time evolution of m in the canonical and microcanonical pictures are displayed, respectively, in Fig. 1 and Fig. 2 for the simplified case $\Gamma = k = 1$. The basic feature of the microcanonical evolution is the existence of a long tail due to the suppression of (relatively) high energy modes.
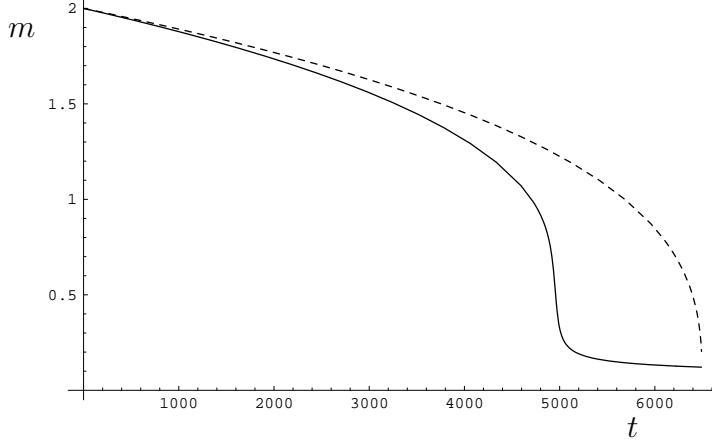
Fig. 2. Time evolution of the radiating black hole in the canonical (dashed line) and microcanonical (continuous line) pictures.

Bearing in mind the above result, we can try and perturb Eq. (16) by assuming an expansion in powers of $1/m$,

$$\dot{m} = -\frac{\kappa_2}{4\,m^2} + \sum_{i\geq 3}\frac{\kappa_i}{m^i}$$

$$x = x_H + \sum_{i\geq 1}\frac{y_i}{m^i} \ ,$$

(25)

where $\kappa_i = \kappa_i(t)$ and $y_i = y_i(t)$. It turns then out that Eqs. (19) and (20) admit as solutions

$$\dot{m} = -\frac{\kappa_2}{4\,m^2} + \frac{\kappa_a}{m^a} + \mathcal{O}\left(\frac{1}{m^{a+1}}\right) \ ,$$

(26)

with $a \geq 3$, κ_a constant, and a corresponding function $x = x(t)$ (whose form is of no relevance here [21]). Again the value of the coefficients in the above equation cannot be fixed within the dynamical approach.

One can fix the values of κ_2 and κ_a in such a way that the luminosity given by Eq. (26) approximates the microcanonical luminosity of Fig. 1 and the history of the mass approximates that obtained from the microcanical picture, at least for $m > 1$. In Fig. 3 we plot $m = m(t)$ for $\kappa_6 = 10^{-4}\,\kappa_2$ and compare it with the microcanonical history. Since we used the large m approximation in order to solve Eqs. (19) and (20), the plot is not reliable for $m < 1$, although it is certainly suggestive that both curves show the presence of a long-living remnant of sub-Planckian mass.

We wish to conclude by remarking that the main result of this letter is given
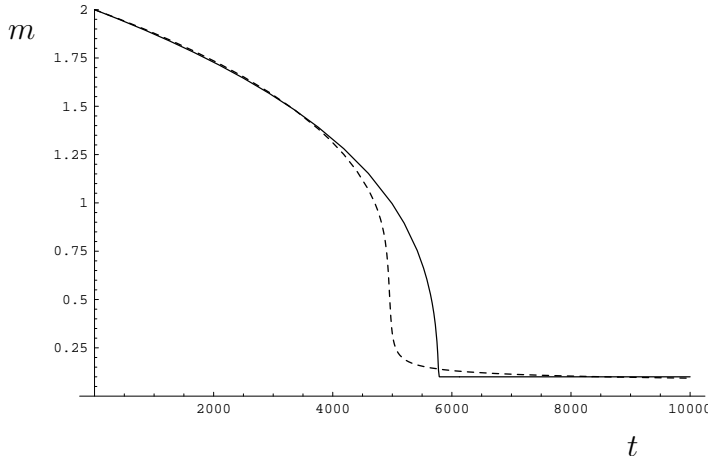
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Fig. 3. Time evolution of the radiating black hole from the Hamiltonian constraint (continuous line) as compared with the microcanonical picture (dashed line) for the case described in the text.

by the set of coupled Eqs. (19) and (20) which govern the dynamics of an evaporating (apparent) horizon and allow one to study the backreaction of the emitted radiation. Unfortunately, they are rather complicated and do not allow for a straightforward analytical treatment. Further, one needs to specify the initial conditions in such a way that relevant physical situations are described. This far, we have just investigated that system of equations by employing the large mass approximation and compared with the results obtained from the statistical mechanics of the Hawking radiation in order to include the backreaction. We hope to carry on a more complete (possibly numerical) analysis and gain a better understanding of the initial conditions in a forthcoming work [21].

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